IB Math HL Y1

Intro to Calculus

Name ______________________________ Date __________

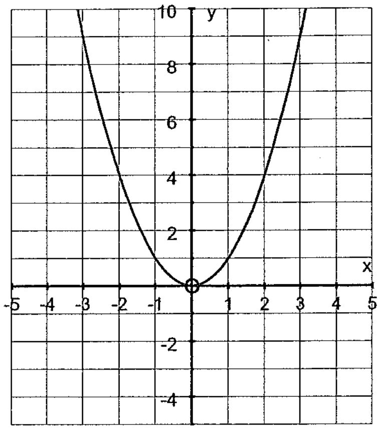
Here's a bold attempt to boil all of calculus down to a **1**-sentence summary:

Calculus is the mathematics of continuous change.

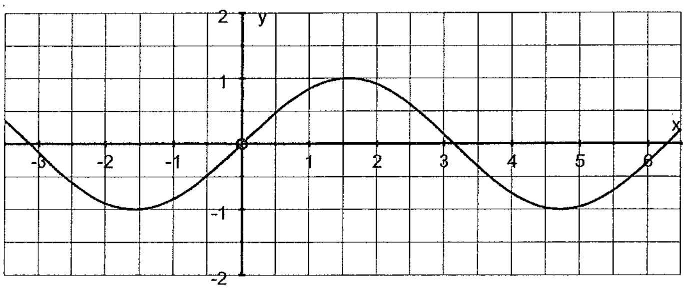
And here are the **2** main problems of calculus (though there are many, MANY variations):

Problem 1 – Find the slope of a curve. [Differential calculus]

1. Estimate the slope of the curve $y = x^2$ at the point $x = 2$. What strategies did you use?

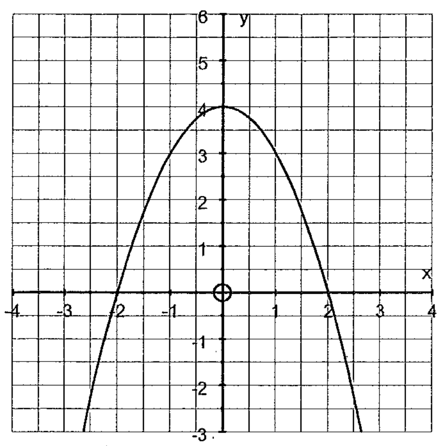


2. Estimate the slope of the curve $y = \sin x$ at a few different points.



Problem 2 – Find the area under a curve. [Integral calculus]

3. Estimate the area bounded by the curve $y = 4 - x^2$ and the x-axis. What strategies did you use?



Also, there are **3** main concepts in calculus (again, with many, MANY variations):

- **Limits**
- **Derivatives**
- **Integrals**

For the next few weeks, we will be studying Differential Calculus, which means we will be focusing on Problem 1.

Average and Instantaneous Velocity

4. The table below shows the progress of a family making a day-long trip by car.

Point	A	B	C	D	E	F	G	H
Time (hr)	0	1.1	2.7	3.3	4.3	5.2	6.5	9.5
Distance (km)	0	98	237	324	431	527	616	729

a) What was the family's average velocity over the whole trip?

b) What was their average velocity between B and C? Between D and G?

c) Can you find the instantaneous velocity from this table? How does a speedometer measure instantaneous velocity?

For any function, the concepts of ***average rate of change*** and ***instantaneous rate of change*** are defined in a similar way to average and instantaneous velocity. The instantaneous rate of change of a function at a point is given by the ***slope of the tangent*** to the graph at that point, and is also known as the ***derivative of the function at that point***.

Notes:

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Limits and Derivatives

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Part 1 – A Quick Look at Limits

1. Using your calculator, graph the function $f(x)=\dfrac{x^2+2x-15}{x-3}$. Describe the behavior of f ***at*** and ***around*** $x=3$.

The mathematical concept that is used to describe this situation formally is that of a ***limit***.

Notes:

2. Evaluate the following limits.

a) $\lim\limits_{x\to 2}\left(x^2+3\right)$

b) $\lim\limits_{x\to 1}\dfrac{x-1}{x^2-1}$

c) $\lim\limits_{x\to 2}\dfrac{\sqrt{x}-\sqrt{2}}{x-2}$

d) $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}$

The concept of a limit can be used to give formal definitions for ***continuity*** and ***continuous functions***.

Notes:

3. Find a value of a such that the piecewise function $f(x)=\begin{cases}2x+3, & x\le 2\\ ax+1, & x>2\end{cases}$ is continuous at $x=2$.

Part 2 – Derivatives from First Principles

Now we return to one of the two main problems of calculus, finding the slope of a curve. Last time we saw that the slope of a curve at a point can be estimated by calculating $\frac{\Delta y}{\Delta x}$, and that the estimate gets better as Δx gets smaller. Now we can't actually let $\Delta x = 0$ (why?), but using the concept of limits, we can let Δx approach 0, and that turns out to be just what we need.

Notes:

When you find the derivative of a function using either of the limits above, we say that you are finding the derivative from ***first principles***. This essentially means that you are not using shortcuts – there are a lot of shortcuts, as we will see starting very soon.

4. Using **Form A**, find the derivative from first principles.

a) $f(x)=2x^2+3$ at $x=3$

b) $g(x)=\frac{9}{x}$ at $x=2$

5. Using **Form B**, find the derivative from first principles.

a) $f(x) = x^2 + 2x$ at $x = 5$

b) $f(x) = x^2 + 2x$, for any value of x

c) $g(x) = x^3$, for any value of x

d) $y = \sqrt{x}$, for any value of x

Part 3 – Extension

6. Use the Binomial Theorem to find the derivative of $f(x) = x^n$, where n is any positive integer.

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Rules for Derivatives

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Today we are going to start looking for rules and shortcuts for derivatives.

Part 1 – Basics

1. These first two are so easy, they hardly count. Answer the following based on your knowledge of the meaning of derivatives.

   a) **Derivative of a constant function** $f(x)=k \Rightarrow f'(x)=$

   b) **Derivative of a linear function** $f(x)=mx+b \Rightarrow f'(x)=$

The first important derivative rule is the **Power Rule**. To find this rule, go back and do the Extension from the previous lesson's handout, then write the result here.

   c) **Derivative of a power (Power Rule)** $f(x)=x^n \Rightarrow f'(x)=$

Notes:

1.

2. Notation

3. Terminology

2. The following rules are also not hard to show.

   a) **Constant multiple rule** $y=k\cdot f(x) \Rightarrow y'=$

   b) **Derivative of a sum/difference** $y=f(x)\pm g(x) \Rightarrow y'=$

With these rules, we can now find the derivative of many basic functions, including all polynomial functions.

3. Differentiate with respect to x.

   a) $f(x)=4x^3$ b) $g(x)=x^4-15x^2+2x$ c) $y=\dfrac{6}{x^2}$

4. Find the slope of the tangent at the indicated point.

   a) $f(x)=x^2+5x-1$, at $x=-1$

   b) $y=3\sqrt{x}$ at $x=4$

5. Find the derivative.

   a) $f(x)=x\sqrt{x}$

   b) $g(x)=\dfrac{4x\left(x^2-3\right)}{x^2}$

   c) $y=\dfrac{2}{\sqrt[3]{x}}$

6. Find the coordinates of the point on the graph of $y=x^2-6x+12$ where the slope is equal to 4.

Part 2 – Product Rule

7. Suppose $f(x)=x^7$ and $g(x)=x^{11}$.

   a) Show that if $y=f(x)\cdot g(x)$, then $y'\neq f'(x)\cdot g'(x)$.

   b) See if you can find the rule for the derivative of a product.

Notes: Alternate notation and proof

8. We previously found the derivative of $f(x) = x\sqrt{x}$. Find this a second way using the Product Rule.

Part 3 – Quotient Rule

Notes:

9. Use the Quotient Rule to differentiate.

 a) $f(x) = \dfrac{x+3}{x-3}$ b) $y = \dfrac{6x^2}{x+1}$ c) $g(x) = \dfrac{\sqrt{x}}{2x+5}$

10. (Optional) We have not yet formally studied the derivatives of trig functions, but we did take an informal look at the slope of $y = \sin x$.

 a) Write a conjecture: $\dfrac{d}{dx}(\sin x) =$

 b) Assume that the conjecture is correct, and differentiate the following.

  i) $y = x^2 \sin x$ ii) $y = \csc x$

 c) Find the values of x at which the graph of $y = \csc x$ has horizontal tangents. Does your answer make sense?

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Chain Rule

Name ____________________________________ Date __________

Part 1 – Investigations

1 a) Can you use the Power Rule to differentiate $y=(4x+1)^2$?

b) Check your answer to (a) by expanding $(4x+1)^2$, then differentiating. Did you get the same answer?

c) What is the derivative of $y=(mx+b)^2$?

d) Make a conjecture about the derivative of $y=(mx+b)^n$. Check by expanding $(2x+1)^3$, then differentiating.

Calculator notes:

2. For this investigation, use the conjecture that $\frac{d}{dx}(\sin x)=\cos x$. What do you think the derivative of $y=\sin 2x$ is? Use what you know about the graphs of sine and cosine functions, and check your ideas using your calculator.

The functions $y=(mx+b)^n$ and $y=\sin 2x$ are both examples of ***composite functions***, i.e. functions of the form $y=f(g(x))$.

3. Identify $f(x)$ and $g(x)$ in the examples above.

   $y=(mx+b)^n$ $\qquad\qquad\qquad$ $y=\sin 2x$

The mathematical rule for finding the derivative of a composite function is called the ***Chain Rule***.

Notes:

Part 2 – Practice

4. Differentiate.

   a) $y=(5-2x)^7$ $\qquad$ b) $f(x)=\sqrt{2x^2+3x}$ $\qquad$ c) $g(x)=\dfrac{5}{(2x-1)^3}$

Terminology note: The term ***gradient*** is often used in place of ***slope***, particularly in British or Australian texts.

5. Find the gradient of the function at the indicated point.

   a) $y=4(2x+3)^5$ at $x=-2$ $\qquad$ b) $f(x)=\dfrac{x}{(3x+2)^4}$ at $x=2$

   [Note: Use 2 rules!]

6. Find the derivative.

a) $y = \sin 4x$ b) $y = \sin^4 x$ c) $y = \sin x^4$

Part 3 – Proofs

7. The Chain Rule can be used to write a proof of the Quotient Rule. See if you can complete the proof, using the first step given below.

Suppose $y = \dfrac{f(x)}{g(x)}$, where $g(x) \neq 0$. Rewrite the function as $y = f(x) \cdot \dfrac{1}{g(x)}$. Then...

8. See if you can write a proof of the Chain Rule, using the given steps to help you.
   - Given $h(x) = f(g(x))$, we want to show that $h'(c) = f'(g(c)) \cdot g'(c)$ for $x = c$.
   - Use Form A of the definition of the derivative: $h'(c) = \lim\limits_{x \to c} \dfrac{h(x) - h(c)}{x - c}$.
   - Do you remember in the proof of the Product Rule, that the key step was to add and subtract a magic expression? In this proof, the key step is to multiply and divide by a (different) magic expression.

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Second Derivatives, Tangents, and Normals
Name ______________________________ Date __________

Part 1 – Second and higher-order derivatives
The derivative of a derivative is called the ***second derivative***.

Notation and other notes:

Higher-order derivatives:

1. If $s(t)$ is a function which gives the position of an object at time *t*:
   a) What does $s'(t)$ represent?
   b) What does $s''(t)$ represent?

2. Find the second derivative.
   a) $f(x)=x^3-\dfrac{3}{x}$
   b) $y=\dfrac{1}{2x+1}$

3. If $f(x)=(2x+5)^n$, what is $f^{(n)}(x)$?

4. Consider the function $f(x)=2x^3+3x^2-36x+5$.
   a) Find $f'(x)$ and determine the interval(s) where $f'(x)>0$.
   b) Find $f''(x)$ and determine the interval(s) where $f''(x)>0$.

Part 2 – Tangents and Normals

5. Find the equation of the tangent to $y = \sqrt{10 - 3x}$ at the point where $x = 3$.

Notice that at the point of tangency, the curve and the tangent have: • The same y-value • The same slope

6. Find the equations of any horizontal tangents to the curve $y = x^3 - 12x + 2$.

The ***normal*** to a curve is perpendicular to the curve at the point of contact.

7. Find the equation of the normal to $y = x^2 - 9x - 12$ at the point where $x = 3$.

Part 3 – More problems

Here's one we did earlier in the year – see if you can solve it using calculus.

8. Find the value of k such that $y = kx - 9$ is tangent to $y = 4x^2 - 8x$. Verify graphically.

9. Find the coordinates of the point(s) where the tangent to $y = x^3 + x + 2$ at (1, 4) meets the curve again.

10. Find the equation of the line which is tangent to $y = x^3 + 2$ and which passes through the origin.

IB Math HL Y1
Derivatives of Exponential and Logarithmic Functions
Name ______________________________ Date __________

Part 1 – Two special derivatives

1. **True** or **False**: $\frac{d}{dx}\left(2^x\right) = x \cdot 2^{x-1}$.

   Find evidence to support your answer, using your calculator or otherwise.

2. Try this mini-investigation:

   Using your graphing calculator, graph $y_1 = a^x$ and $y_2 = \text{nDeriv}\left(y_1, x, x\right)$ on the same set of axes. [Choose a window that allows you to see both graphs clearly.] Try a few different values of a. What type of function does the derivative appear to be? Any other observations? Any particularly interesting values of a?

Notes: Derivative of $y = e^x$

Basic formula: $\frac{d}{dx}\left(e^x\right) =$

Chain Rule version: $\frac{d}{dx}\left(e^u\right) =$

Sketch of proof:

3. Differentiate with respect to x.

   a) $y = 3e^{x^2-2x}$

   b) $y = x^2 e^{-x}$

   c) $y = \frac{e^{2x}}{x}$

4. Use your answer to 3(b) to find the intervals where the derivative of $y = x^2 e^{-x}$ is positive and where it is negative.

5. Another mini-investigation: If $y = \ln x$, what is $\frac{dy}{dx}$?

Use your calculator to find data to help you make a conjecture.

Notes: Derivative of $y = \ln x$	
Basic formula: $\frac{d}{dx}(\ln x) =$	Chain Rule version: $\frac{d}{dx}(\ln u) =$
Proof:	

6. Find the derivative.

a) $y = \ln(2 - x^2)$

b) $y = x^3 \ln x$

c) $y = \ln(\ln(x))$

7. In these problems, be sure to notice how the properties of logarithms can be used.

a) Show that $y = \ln x$ and $y = \ln kx$ have the same derivative. Use log properties to show this a second way.

b) What is the difference between $y=(\ln x)^4$ and $y=\ln x^4$? Find the derivative of each.

c) Differentiate $y=\ln\left(\dfrac{x^2}{(x+2)(x-3)}\right)$ with respect to x. Do you see a difficult way to do this?

An easier way?

8. Find the equation of the tangent to $y=\ln(2x-5)$ at the point where $x = 5$. Use exact values.

In Algebra 2/Trig and Precalculus, students often ask what is special about the number e. I hope you see now that one major part of the answer to this question comes from calculus.

Part 2 – Other bases

9. Write $y=a^x$ in the form $y=e^u$, where u is a function of x. Use this to find the derivative of $y=a^x$.

10. Write $y=\log_a x$ in terms of $\ln x$. Use this to find the derivative of $y=\log_a x$.

Notes:
$\dfrac{d}{dx}\left(a^x\right)=$ $\qquad\qquad\qquad$ $\dfrac{d}{dx}\left(\log_a x\right)=$

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Derivatives as Rates of Change

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Even if you have forgotten every derivative formula you have ever learned, I hope you can still answer this essential question:

What does the derivative of a function at a point tell you?

Write as many different answers as you think appropriate.

So here is one of the main reasons calculus is such a powerful tool for studying the real world – every instantaneous rate of change can be interpreted as a derivative. Or put another way, given any functional relationship between two quantities, you should be able to:

- Explain the meaning of the derivative in context
- Give the units of the derivative

1. Suppose an object is moving along a straight line so that its position at time t is given by the equation $s(t) = 8t - t^3 + 1$, where the position is measured in cm, and t is the time in seconds $(t \geq 0)$.

   a) Find $s'(t)$, and give its meaning and its units.

   b) Find $s''(t)$, and give its meaning and its units.

   c) At $t = 1$, is the object moving forward or backward? Speeding up or slowing down?

Notes:

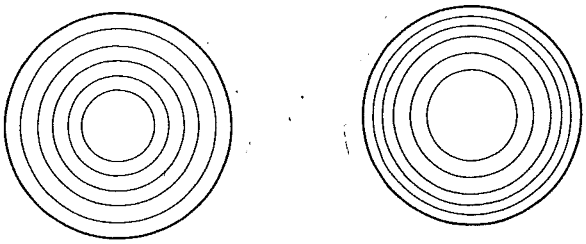
Velocity and acceleration are probably the most frequently encountered rates of change, but there are many others.

2 a) Find the rate of change of the area of a circle with respect to its radius.

b) Evaluate $\frac{dA}{dr}$ at r = 5 and at r = 10.

c) If r is measured in cm and A is measured in cm^2, then what are the units of $\frac{dA}{dr}$?

d) Which of the more appropriate model for tree rings? Can you use derivatives to help you explain your answer?



3. Suppose the function $c(x)$ gives the cost in dollars of producing x units of some manufactured item. Give the meaning and appropriate units of $c'(x)$.

Notes:

The problems on the following pages relate to position, velocity and acceleration. Pay attention to the different information that is given in each problem.

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Trig Derivatives

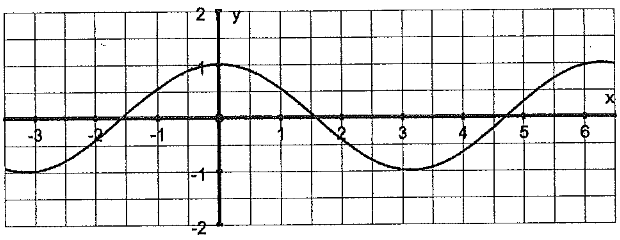
Name ______________________________ Date __________

Part 1 -- Derivatives involving Sine and Cosine

1 a) Do you remember our conjecture about the derivative of $y = \sin x$ (from our very first calculus lesson)?

$$\frac{d}{dx}(\sin x) =$$

b) Use the graph of $y = \cos x$ to estimate the slope at a few points, then make a conjecture about the derivative of $y = \cos x$.



$$\frac{d}{dx}(\cos x) =$$

2. Differentiate the following functions (using the conjectures from Question 1).

a) $y = 3\cos 2x$

b) $y = \sin x^3$

c) $y = \cos^3 4x$

d) $f(x) = x^2 \sin x$

e) $g(x) = \dfrac{\cos 4x}{x}$

f) $y = \ln(\cos x)$

3. Determine the point(s) in the interval $0 \le x < 2\pi$ where the graph of $y = 2\cos x + \sin 2x$ has a horizontal tangent.

Part 2 – Derivatives of Other Trig Functions

4. Use the Quotient Rule to show that $\frac{d}{dx}(\tan x) = \sec^2 x$.

5. Within your group, divide up the remaining trig functions so that you can complete the following:

$$\frac{d}{dx}(\cot x) =$$

$$\frac{d}{dx}(\sec x) =$$

$$\frac{d}{dx}(\csc x) =$$

6. Find the second derivative of $y = \sec x$.

7. Find the equations for the lines that are tangent and normal to the graph of $f(x) = \frac{\tan x}{x}$ at the point where $x = 2$. [You will need your calculator, so do as much as you can with it.]

Part 3 – Applications and Proofs

8. The height of an object attached to a spring is given by the equation $h = \frac{1}{3}\cos 12t - \frac{1}{4}\sin 12t$, where h is measured in inches, and t is measured in seconds.
   a) Find the height and velocity of the object when $t = \frac{\pi}{8}$.
   b) Find the exact value of the maximum displacement.
   c) Find the period P, and the frequency f (number of oscillations per second), given that $f = \frac{1}{P}$.

9. So far, we have only given visual evidence for the derivative of $y = \sin x$. While a complete rigorous proof is beyond the scope of this course, we can do better than we have so far.
   a) Write the expression for the derivative of $y = \sin x$ from first principles.
   b) Use the trig identity for $\sin(A + B)$ to expand the expression from part (a).
   c) Write the expression in the form

   $$\sin x \cdot \lim_{h \to 0}[\quad\quad] + \cos x \cdot \lim_{h \to 0}[\quad\quad],$$

   where the expressions inside the limits are to be determined.
   d) Use tables to estimate the values of the two limits in part (c).

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Implicit Differentiation
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Part 1 – What it is and how it works

Implicit differentiation is a technique for finding the derivative when the equation is not written in the form $y = \ldots$. For example, the equation $x^2 + y^2 = 1$ represents a circle of radius 1, which is not a function, but which clearly has different slopes at different points. In other situations, as with the equation $2x = x^2y^2 + \sin y$, we may not even know what the curve looks like, but after completing this assignment, you should know how to find the slope at any point, even for an equation like this.

Examine the examples below, and look for the pattern.

Expression	Result of differentiation	Expression	Result of differentiation
$3x+5$	3	$3y+5$	$3\cdot\frac{dy}{dx}$
x^2	$2x$	y^2	$2y\cdot\frac{dy}{dx}$
$\sin x$	$\cos x$	$\sin y$	$\cos y\cdot\frac{dy}{dx}$

1. Explain how to differentiate an expression in which the variable is y instead of x. (Can you think why this is necessary?)

2. Use the pattern you observed to differentiate the following expressions:

   a) $4y^3 - 8y^2$ b) $\frac{1}{y^2} + 2x^5 - \sqrt{y}$ c) $e^{x\cos y}$

Part 2 – Problems

3. Consider the equation $xy = 12$. Obviously the point (3, 4) lies on the graph of this equation. We can find the slope of the curve at this point two different ways:

   a) **Method 1:** Solve for y, find y', and evaluate at $x = 3$.

b) **Method 2:** Use implicit differentiation to differentiate both sides of $xy = 12$. Remember to use the Product Rule on the left-hand side. Then solve for $\frac{dy}{dx}$ and evaluate for $x = 3$ and $y = 4$. [Of course, if you didn't get the same answer as in (a), try to figure out what you have to change.]

4. Consider the circle $x^2 + y^2 = 1$.
   a) Use your knowledge of geometry/graphs to answer these questions: Where are the tangents to the circle horizontal? Where are they vertical?

   b) Differentiate both sides of the equation, solve for $\frac{dy}{dx}$, then use it to verify the location of the horizontal and vertical tangents analytically.

5. Find $\frac{dy}{dx}$ if $2x = x^2y^2 + \sin y$ [don't forget the Product Rule!] then find the equation of the line tangent to the curve at $(0, 0)$.

Part 3 – PEP

6. **[IB – short answer, no calculator]**
   Find the gradient of the tangent to $3x^2+4y^2=7$ at the point where $x=1$ and $y>0$.

7. **[AP – extended, no calculator]**
   Consider the curve given by $xy^2-x^3y=6$.
   a) Show that $\frac{dy}{dx}=\frac{3x^2y-y^2}{2xy-x^3}$.
   b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.
   c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

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Inverse Trig Derivatives

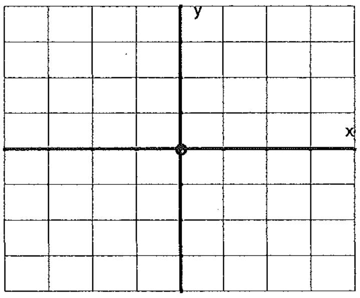
Name ____ Date ____________

Part 1 – Review and Intro

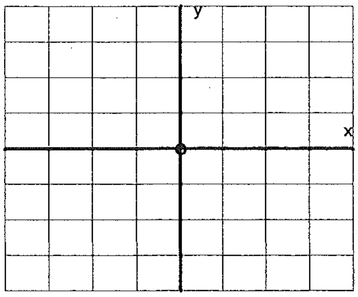
For each of the following:

- Sketch the graph (don't forget to label the axes).
- Indicate the domain and range.
- Write a brief description of the slope – where is it positive/negative, steeper/less steep?

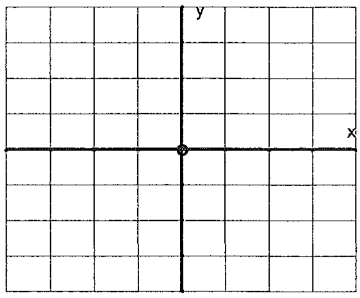
1. $y = \arcsin x$



2. $y = \arccos x$



3. $y = \arctan x$



Part 2 – Formulas and Problems

4. Follow these steps to find the derivative of $y = \arcsin x$.
   - Solve for x.
   - Differentiate both sides (using implicit differentiation).
   - Solve for $\dfrac{dy}{dx}$ (in terms of y).
   - Use a right triangle to find the appropriate trig function in terms of x, not y.
   - Show that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$.

Students are usually surprised that this derivative has no trig functions in its formula. But the derivation shows that there is a trig function of y, just not of x.

5. Use a similar process to find the derivative of $y = \arctan x$.

Similarly, we can show that:

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\operatorname{arc}\cot x) = -\frac{1}{1+x^2} \qquad \frac{d}{dx}(\operatorname{arc}\sec x) = \frac{1}{|x|\sqrt{1-x^2}} \qquad \frac{d}{dx}(\operatorname{arc}\csc x) = -\frac{1}{|x|\sqrt{1-x^2}}$$

The last three are not required for IB; you are not likely to see them on the AP either, though they are technically part of the syllabus.

6. Differentiate.
   a) $f(x) = \sin^{-1} x^2$
   b) $y = x\cos^{-1} 2x$
   c) $y = \arctan e^x$

7. A particle moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \tan^{-1}\sqrt{t}$, where x is measured in cm and t is measured in seconds. Find the exact velocity of the particle when $t = 16$ seconds.

8. Find the equation of the tangent line to the graph of $\arcsin x + \arcsin y = \frac{\pi}{2}$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Part 3 – AP extension – derivatives of inverse functions
It is possible to apply similar reasoning to inverse functions in general.

9. Let $f(x) = x^5 + 2x - 1$. Since the point $(1, 2)$ is on the graph of *f*, the point $(2, 1)$ must be on the graph of $y = f^{-1}(x)$. Use the following steps to find $\left(f^{-1}\right)'(2)$.
   a) Graph $f(x) = x^5 + 2x - 1$. Does it look as if the inverse is a function?
   b) Find $f'(x)$. How could this help you ***prove*** that the inverse is a function?
   c) We know that $f\left(f^{-1}(x)\right) = x$. Differentiate both sides of this equation with respect to x. Be careful to apply the Chain Rule correctly.
   d) Which derivative do we want to solve for? Go ahead and do this.
   e) Can you figure out the value of $\left(f^{-1}\right)'(2)$?

Now try to generalize:

Suppose we know that a function $y = f(x)$ is one-to-one. This means that its graph must pass the ______________________________ and therefore ______ is a function.

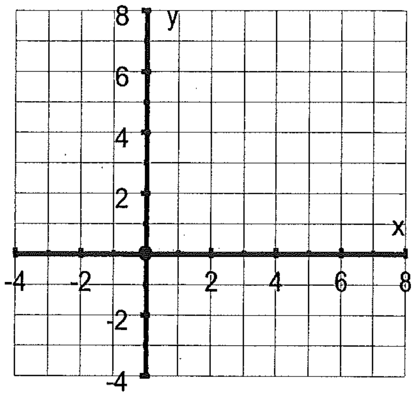
If the slope of $y = f(x)$ at the point $(a, f(a))$ is m, what conclusions can we draw about the slope of $y = f^{-1}(x)$?

IB Math HL Y1
Continuity and Differentiability
Name ______________________________ Date __________

Part 1 – Continuity
We have not made a formal study of what it means for a function to be continuous, but see if you can figure out how to do this question anyway.

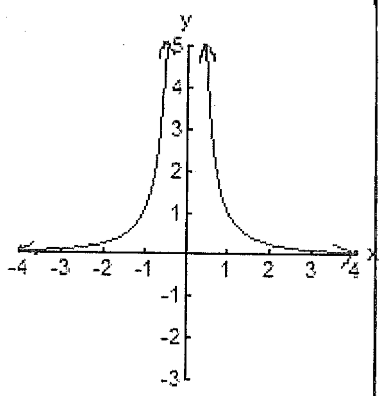
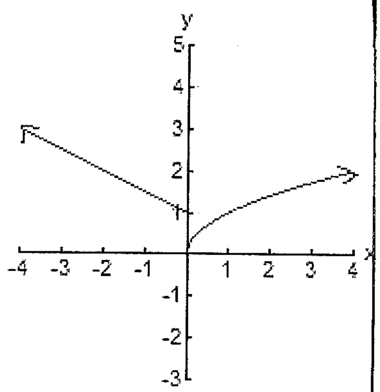
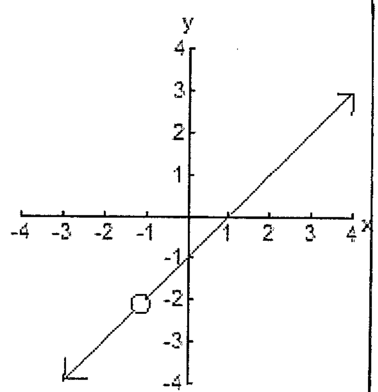
1. Consider the function $f(x)=\begin{cases} x^2-1, & x<2 \\ 2ax, & x\geq 2 \end{cases}$.
   a) Sketch a graph of the function if $a = 1$.

   b) Find the value of a for which the function is continuous at $x = 2$.



Informally, a function is ***continuous*** if its graph can be drawn without picking up your pen or pencil. (We will take a look at the formal definition in just a bit.)

There are several types of discontinuity, as shown below:

Graphs of Discontinuous Functions		
Infinite Discontinuity $f(x)=\dfrac{1}{x^2}$	Jump Discontinuity $f(x)=\begin{cases} -0.5x+1, x<0 \\ \sqrt{x}, \quad x>0 \end{cases}$	Point Discontinuity $f(x)=\dfrac{x^2-1}{x+1}$
		

2 a) What feature does a graph have at an infinite discontinuity?

b) What feature does a graph have at a point discontinuity?

3. The function $f(x)=\dfrac{x-2}{x^2-4}$ has two discontinuities. Find the x-values where these discontinuities occur, and classify them.

4 a) Suppose we are working with a rational function. Explain how to distinguish ***algebraically*** between infinite discontinuities and point discontinuities.

b) Without graphing, find the x-values where the discontinuities of $f(x)=\dfrac{2x^2+5x-3}{2x^2+7x+3}$ occur, and classify them.

5. Consider the function $f(x)=\dfrac{\sin x}{x}$. Does this function have any discontinuities? If so, where, and what type? [How does this relate to other ideas we have talked about recently?]

This example points out the need to have a more general framework for describing the continuity/discontinuity of functions. Here is the formal definition for when a function f is continuous at $x = a$ – it's quite a bit more complicated than the informal definition, even though it means exactly the same thing.

A function $y=f(x)$ is continuous at $x=a$ if $\lim_{x\to a} f(x)=f(a)$. In order for this to be true:

6. Go back to Question 1(a), and state $\lim_{x\to 2-} f(x)$ and $\lim_{x\to 2+} f(x)$.

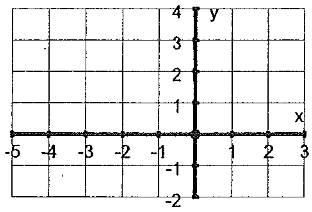
Terminology note: Infinite and jump discontinuities are also known as ***non-removable*** discontinuities, while point discontinuities are also known as ***removable*** discontinuities.

7. Why do you think the terms removable and non-removable are used? Go back to questions 1-5 and label all discontinuities as either removable or non-removable.

Part 2 – Differentiability

A function is ***differentiable*** at $x = a$ if $f'(a)$ exists.

8. Sketch the graph of $f(x) = |x+2|$.



   a) What is $f'(-3)$?

      $f'(1)$?

   b) What can you say about $f'(-2)$?

9. Investigate the slope of $f(x) = x^{1/3}$ at $x = 0$ using ***symmetric difference quotients***, i.e. slope expressions of the form $\dfrac{f(a+h)-f(a-h)}{2h}$. In general, h should be a small positive number, and in this case $a = 0$.

h	$\dfrac{f(a+h)-f(a-h)}{2h}$

What do you observe?

10. Do the same thing with $f(x) = x^{2/3}$, but use ***left and right difference quotients***.

h	$\dfrac{f(a+h)-f(a)}{h}$	$\dfrac{f(a)-f(a-h)}{h}$

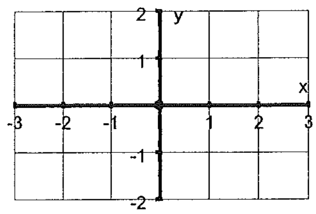
What do you observe?

Difference quotients are used to estimate derivatives if you don't have the derivative as a function (for example, if you are working with data), but difference quotients can be deceiving also.

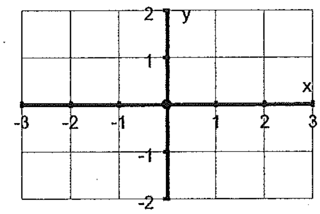
11. Use your calculator to find the derivative of $f(x)=|x|$ at $x=0$. What happens? [Sean, see me for a substitute problem to do by hand!]

You should know the following ways that a function can fail to be differentiable, and an example of each.

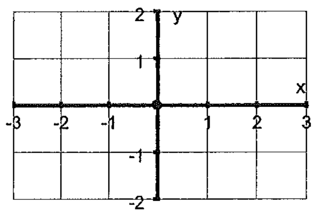
a) [*corner*]



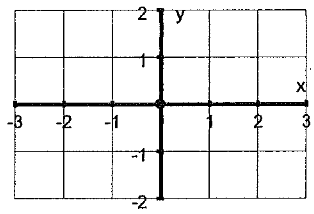
b) [*cusp*]



c) [***vertical tangent***]



d) [***discontinuity***]



12. Given $f(x)=\begin{cases} 3x+b, & x<1 \\ x^2-ax+8, & x\geq 1 \end{cases}$.

Find a and b so that f is continuous and differentiable at $x=1$.